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SOLUTIONS OF THE OPERATOR-VALUED INTEGRATED CAUCHY FUNCTIONAL EQUATION

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ABSTRACT. Let G be a separable, metrizable locally compact abelian group and let σ be a vector measure on G taking values in the centre of a von Neumann algebra $\mathcal{A}$. Given an $\mathcal{A}$-valued measure μ on G, we define the convolution $\mu * \sigma$ and study the equation $\mu = \mu * \sigma$, using Choquet's integral representation theory as in [7] where the same equation for scalar measures was studied.

KEYWORDS: *Convolution equation, locally compact group, exponential function, operator-valued measure, von Neumann algebra, conditional expectation, Choquet's integral representation.*

AMS SUBJECT CLASSIFICATION: Primary 43A05; Secondary 46G10, 47C15, 46A55, 46L10, 47A56.

1. INTRODUCTION

Given a locally compact group G and a Borel measure σ on G, the integrated Cauchy functional equation

$$f(x) = \int_G f(x-y)\,\mathrm{d}\sigma(y) \qquad (x \in G)$$

has been studied by many authors (cf. [5], [6], [7], [9], [10], [14], [16], [18], [20]) and the real or complex-valued solutions f have been characterized under various assumptions and with diverse techniques using devices such as Fourier transform (e.g., [13], [20]), Martingales [9] and Choquet's integral representation theory (e.g., [5], [7]). In particular, Choquet and Deny [5] proved that if G is separable, metrizable and abelian, and if σ is a probability measure such that $\operatorname{supp}\sigma$ generates G, then the bounded solutions are constant functions. If both f and σ are nonnegative, then Deny [7] showed, as a development of [5], that f can be represented as an integral of the exponential functions g on G (i.e., $g(x+y) = g(x)g(y)$) satisfying $\int_G g(-y)\mathrm{d}\sigma(y) = 1$. In fact, Deny considered the more general convolution equation

$$\mu = \mu * \sigma \tag{1.1}$$

where μ is a nonnegative Borel measure on G. The solutions μ are of the form $\mu = f\lambda$ where λ is the Haar measure on G and f is as above.

The integrated Cauchy functional equation has many important applications (cf. [1], [10], [12], [18]) and it is natural and desirable to seek *vector-valued* solutions f (or μ) of the equation. The case that f is an $\mathbf{R}^n$-valued function and σ is a matrix-valued measure has been considered in [15]. In such case positivity is defined to be coordinatewise positive, and the probability measure used in the scalar case is replaced by a positive measure σ so that $\sigma(G)$ is a Markov matrix (i.e., the sum of each row is 1). The vector-valued theorem thus extended is used to solve a vector-valued renewal equation, which is in turn used to study some class of self-similar fractal measures.

Let G be a separable, locally compact metrizable abelian group, and $\mathcal{A}$ a von Neumann algebra of bounded operators on a (complex) separable Hilbert space H with centre Z. In this paper, we study equation (1.1) where σ is a given positive Z-valued measure and μ is a positive *extended* $\mathcal{A}$-valued measure on G. The basic difference of this consideration from [15] is that the positivity here refers to the positive definiteness of an operator. The extended $\mathcal{A}$-valued measure (including the '∞' in the range (Section 3)) is used because we want to include the unbounded solutions also. Following Bartle [2], we define the bilinear vector integral $\int_G f\,\mathrm{d}\sigma$, for an $\mathcal{A}$-valued function f, as an element in $\mathcal{A}$. This is used to define the convolution $\mu * \sigma$. Our main results are the following extension of Deny's theorem:

THEOREM 4.11. *Under the above assumption, let H_σ be the cone of positive solutions of* (1.1) *and ∂H_σ the extremal elements in H_σ. Then $\mu \in \partial H_\sigma$ if and only if* $\mathrm{d}\mu(x) = cpg(x)\mathrm{d}\lambda(x)$ *where $c > 0$, λ is the Haar measure on G, p is a minimal projection in $\mathcal{A}$, and $g : G \to (0,\infty)$ satisfies*

$$g(x+y) = g(x)g(y) \quad \text{and} \quad p = p\left(\int_G g(-y)\,\mathrm{d}\sigma(y)\right).$$

THEOREM 5.6. *If in addition, $\mathcal{A}$ is atomic and the solution μ is also a positive extended $T(H)$-valued measure ($T(H)$ denotes the trace-class operators on H), then μ is a 'mixture' of the above extremal solutions in the sense that there is* *a probability measure* $\mathbf{P}$ *on* H_σ *supported by a Borel subset* B *of* $\partial H_\sigma \bigcup \{0\}$ *such that*

$$\mu = \int_B \nu \, \mathrm{d}\mathbf{P}(\nu).$$

That σ takes values in the centre Z of $\mathcal{A}$ is crucial in the proof of Theorem 4.11. First, if ρ is a pure state of $\mathcal{A}$, then $\rho(az) = \rho(a)\rho(z)$ for all $a \in \mathcal{A}$ and $z \in Z$ (Lemma 2.1). This allows us to reduce the equation into scalar form so that Deny's technique is applicable. Second, since $\mathcal{A}$ acts on a separable Hilbert space, $\mathcal{A}$ has a faithful normal state and there is a faithful contractive projection from $\mathcal{A}$ onto Z which is essential for constructing the exponential function g in the theorem (Lemma 4.9). The extremal solution in Theorem 4.11 may not exist in general and the atomic assumption on $\mathcal{A}$ in Theorem 5.6 guarantees such existence. The additional assumption on the measure μ in Theorem 5.6 implies that μ is contained in a *cap* of a cone containing H_σ so that Choquet's integral representation applies.

2. PRELIMINARIES

Recall that a C^*-*algebra* $\mathcal{A}$ is a norm-closed $*$-subalgebra of the algebra $B(H)$ of all bounded operators on a Hilbert space H. We call $\mathcal{A}$ a *von Neumann algebra* (or W*-*algebra*) if it has a (unique) predual $\mathcal{A}_*$; in this case $\mathcal{A}$ contains an identity and we can assume without loss of generality that it is the identity operator $\mathbf{I}$ in $B(H)$. We denote by $\mathcal{A}_{sa} = \{a \in \mathcal{A} : a^* = a\}$ the *self-adjoint part* of $\mathcal{A}$, and by $\mathcal{A}_+ = \{a^*a : a \in \mathcal{A}\}$ the cone of positive operators in $\mathcal{A}$ which defines a partial ordering $\leqslant$ in $\mathcal{A}_{sa}$. We refer to [22] for the basics of operator algebras.

A *state* of a C^*-*algebra* $\mathcal{A}$ is a complex linear functional ρ on $\mathcal{A}$ such that $\|\rho\| = 1$ and $\rho \geqslant 0$ where the latter means $\rho(a^*a) \geqslant 0$ for all $a \in \mathcal{A}$. A state ρ of $\mathcal{A}$ is called *pure* if for any state ψ satisfying $\alpha\psi \leqslant \rho$ for some $\alpha > 0$, then one must have $\psi = \rho$. We note that the pure states of $\mathcal{A}$ separate points of $\mathcal{A}$ in that given $a, b \in \mathcal{A}_{sa}$, then $a \leqslant b$ if and only if $\rho(a) \leqslant \rho(b)$ for every pure state ρ of $\mathcal{A}$. If $\mathcal{A}$ is a von Neumann algebra and if $\rho \in \mathcal{A}_*$ is a state of $\mathcal{A}$, then ρ is called a *normal state* of $\mathcal{A}$. Normal states of $\mathcal{A}$ also separate points in $\mathcal{A}$.

If $\mathcal{A}$ is a commutative C^*-algebra, then a state ρ of $\mathcal{A}$ is pure if and only if it is multiplicative, i.e., $\rho(ab) = \rho(a)\rho(b)$ for all $a, b \in \mathcal{A}$. We will make frequent use of the following result of Størmer in ([21], Theorem 3.1) and we include the proof here for completeness.

LEMMA 2.1. *Let $\mathcal{A}$ be a C^*-algebra containing the identity I and with centre Z. Let ρ be a pure state of $\mathcal{A}$.*

Then the restriction $\rho|_Z$ is a pure state of Z and furthermore,

$$\rho(az) = \rho(a)\rho(z) \tag{2.1}$$

for all $a \in \mathcal{A}$ and $z \in Z$.

Proof. Since $Z = Z_{sa} + iZ_{sa}$, we need only consider $z \in Z_{sa}$. Without loss of generality we assume that $\|z\| < \frac{1}{2}$ say. Then

$$|\rho(z)| \leqslant \|\rho\| \cdot \|z\| < \frac{1}{2} \qquad \text{and} \qquad \frac{1}{2} - \rho(z) > 0.$$

Let $\alpha = 1/2 - \rho(z)$ and define $\psi : \mathcal{A} \to \mathbf{C}$ by

$$\psi(a) = \alpha^{-1}\rho\left(a\left(\frac{I}{2} - z\right)\right), \qquad a \in \mathcal{A}.$$

Then ψ is a state of $\mathcal{A}$ and $\alpha\psi \leqslant \rho$. As ρ is pure, we have $\psi = \rho$ which gives, for $a \in \mathcal{A}$, $\rho(a) = \alpha^{-1}\rho(\frac{a}{2} - az)$, yielding $\rho(az) = \rho(a)\rho(z)$. ∎

REMARK. Every pure state on Z extends to a pure state on $\mathcal{A}$. This fact will be used later.

Throughout G will always denote a locally compact abelian group, $\mathcal{B}$ the σ-algebra of Borel sets in G, and $\mathcal{A}$ is a von Neumann algebra with centre Z. Let $\sigma : \mathcal{B} \to Z$ be a (norm) countably additive *positive* measure, that is, $\sigma(E) \geqslant 0$ in Z for all $E \in \mathcal{B}$. For the natural bilinear map

$$(a, z) \in \mathcal{A} \times Z \longmapsto az \in \mathcal{A},$$

we define the *semi-variation* of σ on any $E \in \mathcal{B}$ as

$$\|\sigma\|(E) = \sup\left\|\sum a_i\sigma(E_i)\right\|$$

where the supremum is taken over all $a_i \in \mathcal{A}$ with $\|a_i\| \leqslant 1$ and all partitions $\{E_i\}$ of E [2]. Since $\mathcal{A}$ contains identity and σ is positive, $\|\sigma\|(E)$ equals $\|\sigma(E)\|$. In particular σ has finite semi-variation by taking $E = G$. We can also define, as in [2], a *σ-integrable* function $f : G \to \mathcal{A}$ and the so-called *bilinear vector integral* $\int_E f\, d\sigma$ for $E \in \mathcal{B}$. For convenience and completeness, we give below an *ad hoc* construction of the integral which is equivalent to Bartle's integral.

First, if $f : G \to \mathcal{A}$ is a simple function, say, $f = \sum_i a_i \chi_{E_i}$ with $a_i \in \mathcal{A}$ and $E_i \in \mathcal{B}$, we define

$$\int_E f \, d\sigma = \sum_i a_i \sigma(E \bigcap E_i)$$

for $E \in \mathcal{B}$. Since $\|f(x)\| = \sum_i \|a_i\| \chi_{E_i}(x)$ for $x \in G$, we have

$$\Big\| \int_E f \, d\sigma \Big\| \leqslant \Big\| \int_E \|f(x)\| \, d\sigma(x) \Big\| \leqslant (\sup_i \|a_i\|) \|\sigma(E)\|.$$

A function $f : G \to \mathcal{A}$ is said to be *σ-integrable* if it satisfies the following two conditions:

(i) There is a sequence $\{f_n\}$ of simple functions on G such that $\lim_{n\to\infty} \|f_n(x) - f(x)\| = 0$ for each x in some $E \in \mathcal{B}$ with $\|\sigma(G \setminus E)\| = 0$;

(ii) The sequence $\left\{ \int_E f_n \, d\sigma \right\}$ is norm convergent in $\mathcal{A}$ for every $E \in \mathcal{B}$.

We define, as usual,

$$\int_E f \, d\sigma = \lim_{n\to\infty} \int_E f_n \, d\sigma \in \mathcal{A}.$$

It follows from Lemma 2.1 that if f is σ-integrable, then for any pure state ρ of $\mathcal{A}$,

$$\rho \left(\int_E f \, d\sigma \right) = \int_E (\rho f) \, d\rho\sigma \qquad E \in \mathcal{B} \tag{2.2}$$

where $\rho f = \rho \circ f$ and $\rho\sigma = \rho \circ \sigma$.

We note that for every $t \in \mathcal{A}_{sa}$, $\|t\| \leqslant r$ if and only if $-r\mathbf{I} \leqslant t \leqslant r\mathbf{I}$. Given a function $f : G \to \mathcal{A}_{sa}$ such that there is a sequence $f_n : G \to \mathcal{A}_{sa}$ of simple functions with $\lim_{n\to\infty} \|f_n(x) - f(x)\| = 0$ for every $x \in G$, then we have

$$\{x \in G : \|f(x)\| \leqslant r\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{ x \in G : -\left(r + \frac{1}{k}\right)\mathbf{I} \leqslant f_n(x) \leqslant \left(r + \frac{1}{k}\right)\mathbf{I} \right\} \in \mathcal{B},$$

and we have the following version of Egorov's theorem.

LEMMA 2.2. *Let $f : G \to \mathcal{A}_{sa}$ and let $\{f_n\}$ be a sequence of simple functions such that $\lim_{n\to\infty} \|f_n(x) - f(x)\| = 0$ for each $x \in G$. Then $f_n \to f$ almost uniformly, i.e., for each $\varepsilon > 0$, there exists $E \in \mathcal{B}$ such that $\|\sigma(G \setminus E)\| < \varepsilon$ and $\sup_{x \in E} \|f_n(x) - f(x)\| \to 0$ as $n \to \infty$.*

By analogous proof as in the scalar case we have:

LEMMA 2.3. *Let $f : G \to \mathcal{A}_+$ be such that for $r \in \mathbf{R}$, the sets $\{x \in G : f(x) \leqslant r\mathbf{I}\}$ and $\{x \in G : r\mathbf{I} \leqslant f(x)\}$ are in $\mathcal{B}$. Then there is an increasing sequence of simple functions $f_n : G \to \mathcal{A}_+$ such that $f_n \leqslant f$ and $\lim_{n\to\infty} \|f_n(x) - f(x)\| = 0$ for each $x \in G$. Moreover, if f is bounded, then f is σ-integrable and $\int_E f \, d\sigma = \lim_{n\to\infty} \int_E f_n \, d\sigma$ for every $E \in \mathcal{B}$.*

Later on we will also use the vector integral in which the roles of $\mathcal{A}$ and Z are interchanged, i.e., the vector integral $\int_E g \, d\mu$ with respect to the bilinear map

$$(z, a) \in Z \times \mathcal{A} \to za \in \mathcal{A}$$

where $\mu : \mathcal{B} \to \mathcal{A}$ is a positive countably additive measure and $g : G \to Z$ is a μ-integrable function. As in (2.2), we also have $\rho(\int_E f \, d\sigma) = \int_E (\rho f) \, d\rho\sigma$ for every pure state ρ of $\mathcal{A}$.

To conclude this section we remark that if $\mathcal{A}$ is the algebra of $n \times n$ matrices, then the center Z is the scalar multiples of the identity matrix $\mathbf{I}$. Coordinatewise the $\mathcal{A}$-valued equation $f(x) = \int_G f(x - y) \, d\sigma(y)$ becomes

$$f_{ij}(x) = \int_G f_{ij}(x - y) \, d\tau(y).$$

where τ is a scalar measure. The matrix extension of the Choquet-Deny [5] theorem (i.e. the case f is bounded) is easily achieved by characterizing each f_{ij} separately. However for the extension of Deny's theorem [7], the reader should be cautioned that although f is assumed to be positive-definite-valued, it does not imply that each f_{ij} is positive and hence the scalar Deny theorem can not be applied coordinatewise to characterize the solutions of the above equation. Further even if the general solution of each f_{ij} can be obtained, simply putting these f_{ij} together need not form a positive definite matrix-valued solution f of the integrated Cauchy functional equations.

We note that any finite dimensional von Neumann algebra is a finite direct sum of matrix algebras, and in this case the convolution equation can be reduced coordinatewise as above. To illustrate the idea, we give the following simple example with a nontrivial center Z:

Let $M_2(\mathbf{C})$ be the algebra of 2×2 complex matrices and let ℓ_2^∞ be the 2-dimensional commutative von Neumann algebra, i.e., $\mathbf{C}^2$ equipped with the ℓ^∞–

norm. Let $\mathcal{A} = M_2(\mathbb{C}) \otimes \ell_2^\infty$ with centre $Z = \mathbf{I} \otimes \ell_2^\infty$. Then a function $f : G \to \mathcal{A}$ can be represented as follows:

$$f(x) = [f_{ij}(x)] = \begin{bmatrix} f_{11}(x) & 0 & f_{13}(x) & 0 \\ 0 & f_{22}(x) & 0 & f_{24}(x) \\ f_{31}(x) & 0 & f_{33}(x) & 0 \\ 0 & f_{42}(x) & 0 & f_{44}(x) \end{bmatrix}$$

where $f_{ij} : G \to \mathbb{C}$. A $Z-$valued measure σ on G can be written as

$$\sigma = \begin{bmatrix} \sigma_{11} & & & \\ & \sigma_{22} & & \\ & & \sigma_{33} & \\ & & & \sigma_{44} \end{bmatrix}$$

where $\sigma_{11} = \sigma_{33}$ and $\sigma_{22} = \sigma_{44}$ are complex-valued measures on G. In this case, the operator-valued equation

$$f(x) = \int_G f(x-y)\, \mathrm{d}\sigma(y)$$

implies the following simultaneous equations:

$$f_{ij}(x) = \int_G f_{ij}(x-y)\, \mathrm{d}\sigma_{jj}(y)$$

and the above remarks apply to these scalar equations as well.

3. OPERATOR-VALUED MEASURES

We will further assume that G is separable metrizable so that it is $\sigma-$compact: $G = \bigcup_{n=1}^{\infty} G_n$ where each G_n is a compact subset of G and $G_n \subset G_{n+1}^\circ$. Following [3], we let $K(G, \mathbb{R})$ be the real vector space of real continuous functions on G with compact support. We equip $K(G, \mathbb{R})$ with the pointwise ordering and with the inductive topology as in ([3], p.66, [4], p.13). The dual $K(G, \mathbb{R})^*$, consisting of *continuous* linear functionals, is precisely the set of regular Borel measures (Radon measures) on G, and the positive cone $K(G, \mathbb{R})_+^*$ the positive ones ([3], Section 11). Given a net $\{\mu_\alpha\}$ in $K(G, \mathbb{R})^*$, we say that $\{\mu_\alpha\}$ converges to $\mu \in K(G, \mathbb{R})^*$ *vaguely* if $\{\mu_\alpha\}$ converges to μ in the w^*-topology, that is, $\mu_\alpha(f) = \int_G f\, \mathrm{d}\mu_\alpha \to \mu(f)$ for all $f \in K(G, \mathbb{R})$.

More generally, if X is a real Banach space partially ordered by a cone X_+, we let $K(G,X)$ be the real vector space of continuous functions from G to X with compact support, and let $K(G_n,X)$ be its subspace consisting functions with supports in G_n. With the supremum norm, $K(G_n,X)$ is a Banach space and its dual $K(G_n,X)^*$ identifies with the space $M(G_n,X^*)$ of X^*-valued Borel measures on G_n with bounded total variation. Since $K(G,X)$ is the inductive limit of the increasing sequence $\{K(G_n,X)\}_{n=1}^{\infty}$ of spaces, we can equip $K(G,X)$ with the inductive topology as in ([4], p.13) so that the w^*-topology on $K(G,X^*)$ is the product topology defined by $\{M(G_n,X^*)\}_{n=1}^{\infty}$. For $f,h \in K(G,X)$, we write $f \leqslant h$ to mean that $f(x) \leqslant h(x)$ in X, for every $x \in G$.

Let $\mathcal{A}$ be a von Neumann algebra as before, and henceforth let $\mathcal{A}_*$ be the (real) predual of the real Banach space $\mathcal{A}_{sa}$. Then the cone $(\mathcal{A}_*)_+$ is in duality with the cone $\mathcal{A}_+$ in $\mathcal{A}_{sa}$. For $t \in \mathcal{A}_*$, $|t|$ is defined as in ([22], III 4.3), and satisfies

$$t^{\pm} = \frac{1}{2}(|t| \pm t) \quad \text{and} \quad \|t^{\pm}\| = \frac{1}{2}\big\||t| \pm t\big\| \leqslant \|t\|.$$

If $\{t_n\}$ is a sequence in $\mathcal{A}_*$ norm-convergent to $t \in \mathcal{A}_*$, then $\{|t_n|\}$ converges to $|t|$ in norm ([22], p.145). Therefore using $t = t^+ - t^-$, each $f \in K(G,\mathcal{A}_*)$ can be decomposed as $f = f^+ - f^-$ where $f^{\pm} \in K(G,\mathcal{A}_*)$ are positive.

We thank Professor C. Lennard for the proof of the following result.

LEMMA 3.1. *Let $\varphi : K(G,\mathcal{A}_*) \to \mathbf{R}$ be a positive linear functional. Then φ is continuous, that is, $\varphi \in K(G,\mathcal{A}_*)^*$.*

Proof. It suffices to prove that the restrictions $\varphi_n = \varphi|K(G_n,\mathcal{A}_*)$ are continuous. Suppose some φ_n is not continuous. Then there is a sequence $\{f_m\}$ in $K(G_n,\mathcal{A}_*)$ such that $\|f_m\| \leqslant 1$ and $|\varphi_n(f_m)| \to \infty$ as $m \to \infty$. Since

$$|\varphi_n(f_m)| \leqslant \varphi_n(f_m^+) + \varphi_n(f_m^-),$$

we may assume $\varphi_n(f_m^+) \to \infty$, say. By the above remarks, we have $\|f_m^+\| \leqslant \|f_m\| \leqslant 1$. Choose a subsequence $\{f_k\}$ of $\{f_m\}$ such that $\varphi_n(f_k^+) \geqslant 2^k$ for all $k \geqslant 1$. Then $\sum_{k=1}^{\infty} \frac{1}{2^k} f_k^+ \in K(G_n,\mathcal{A}_*)$ and hence

$$\varphi_n\Big(\sum_{k=1}^{\infty} \frac{1}{2^k} f_k^+\Big) \geqslant \varphi_n\Big(\sum_{k=1}^{N} \frac{1}{2^k} f_k^+\Big) = \sum_{k=1}^{N} \frac{1}{2^k}\varphi_n(f_k^+) \geqslant N \quad \text{for all} \quad N \in \mathbf{N},$$

which is impossible. So φ_n is continuous for all n. ∎

Let X be a real partially ordered Banach space with a monotone closed cone X_+ (i.e., every bounded increasing sequence in X_+ converges). By an *extended X_+-valued measure* on G we mean a countably additive function

$$\mu : \mathcal{B} \to X_+ \bigcup \{\infty\}$$

such that $\mu(K) \in X_+$ for every compact subset K of G, where the symbol $\infty \notin X_+$ satisfies

$$0 \cdot \infty = 0$$
$$\infty + \infty = \infty$$
$$r \cdot \infty = \infty$$
$$t + \infty = \infty + t = \infty$$
$$t \leqslant \infty$$

for $r > 0$ and $t \in X_+$. We write $\sum_{n=1}^{\infty} x_n = \infty$ if the series $\sum_{n=1}^{\infty} x_n$ diverges in X_+. We will denote this class of measures by $M(G, X_+)$. Given $\mu, \nu \in M(G, X_+)$, we write $\mu \leqslant \nu$ to mean that $\mu(E) \leqslant \nu(E)$ for all $E \in \mathcal{B}$.

Given $\mu \in M(G, \mathcal{A}_+)$ and a state ρ of $\mathcal{A}$, we define $\rho\mu : \mathcal{B} \to [0, \infty]$ by $\rho\mu(E) = \lim_{n\to\infty} \rho\mu(E \bigcap G_n)$, then $\rho\mu$ a regular Borel measure on G ([19], Theorem 2.18). It follows that for any $\mu, \nu \in M(G, \mathcal{A}_+)$, we have $\mu \leqslant \nu$ whenever $\mu(K) \leqslant \nu(K)$ for every compact set $K \subset G$, since the latter implies that for every state ρ of $\mathcal{A}$, $\rho\mu \leqslant \rho\nu$ by regularity. Note that the states separate points of $\mathcal{A}$.

A von Neumann algebra $\mathcal{A}$ is called *σ−finite* ([22], p.78) if there exists a normal state $\kappa \in \mathcal{A}_*$ which is *faithful* in that whenever $a \in \mathcal{A}_+$ and $\kappa(a) = 0$, then $a = 0$. We note that every von Neumann algebra acting on a *separable* Hilbert space H has a separable predual and is σ-finite, and that a commutative σ-finite von Neumann algebra is just an $L^\infty(\nu)$ where ν is a σ-finite complex measure.

LEMMA 3.2. *Let $\mathcal{A}$ be a σ-finite von Neumann algebra (with faithful normal state κ). Then there is a one-one correspondence between $M(G, \mathcal{A}_+)$ and the positive linear functionals on $K(G, \mathcal{A}_*)$.*

Proof. Given $\mu \in M(G, \mathcal{A}_+)$, let $\mu_n : \mathcal{B}_{G_n} \to \mathcal{A}_+$ be the restriction of μ to G_n. Then there exist positive funcitonals $\varphi_n : K(G_n, \mathcal{A}_*) \to \mathbf{R}$ such that $\varphi_n(f) = \int_{G_n} f \, \mathrm{d}\mu_n$ for $f \in K(G_n, \mathcal{A}_*)$ where the (bilinear) vector integral is defined as in [2] using the bilinear map

$$(f, \psi) \in K(G_n, \mathcal{A}_*) \times K(G_n, \mathcal{A}_*)^* \to \psi(f) \in \mathbf{R}.$$

The corresponding positive functional $\varphi : K(G, \mathcal{A}_*) \to \mathbf{R}$ is then given by $\varphi = \lim_{n\to\infty} \varphi_n$.

Conversely, let $\varphi : K(G, \mathcal{A}_*) \to \mathbf{R}$ be a positive functional and let φ_n be its restrictions to $K(G_n, \mathcal{A}_*)$. Then there exists, for each n, a measure $\mu_n \in M(G_n, \mathcal{A}_+)$ such that $\varphi_n(f) = \int_{G_n} f \, d\mu$ for all $f \in K(G_n, \mathcal{A}_*)$. Note that $\mu_m = \mu_n$ on G_n for $m \leqslant n$.

Now we are going to define $\mu : \mathcal{B} \to \mathcal{A}_+ \bigcup \{\infty\}$ associated with φ. For each $E \in \mathcal{B}$, the sequence $\{\mu_n(E \bigcap G_n)\}_{n=1}^{\infty}$ is increasing in $\mathcal{A}_+$, by positivity of μ_n. Since $\mathcal{A}_{sa}$ is monotone closed in the sense of ([22], p.137), we can define

$$\mu(E) = \begin{cases} \sup_n \mu_n(E \bigcap G_n) = s\text{-}\lim_{n\to\infty} \mu_n(E \bigcap G_n), & \text{if } \{\|\mu_n(E \bigcap G_n)\|\}_{n=1}^{\infty} \text{ is bounded,} \\ \infty & \text{otherwise,} \end{cases}$$

where 's-lim' denotes the limit in the strong operator topology on $\mathcal{A} \subset B(H)$. Evidently μ is finitely additive, and since strong-operator convergence implies $\sigma(\mathcal{A}, \mathcal{A}_*)$-convergence, we know that the scalar measure $\rho\mu$ is countably additive for every normal state $\rho \in \mathcal{A}_*$. We show that μ is indeed countably additive which will complete the proof. Let $E = \bigcup_{k=1}^{\infty} E_k$ be a disjoint union of Borel sets in G.

Case (i): If $\|\mu_n(E \bigcap G_n)\| \to \infty$ as $n \to \infty$, then $\mu(E) = \infty$ by definition. On the other hand, by the uniform boundedness principle, there exists a normal state ρ in $\mathcal{A}_*$ such that $\rho\mu_n(E \bigcap G_n) \to \infty$. It follows that

$$\sum_{k=1}^{\infty} \rho\mu(E_k) = \rho\mu(E) = \lim_{n\to\infty} \rho\mu_n(E \bigcap G_n) = \infty.$$

Therefore $\sum_{k=1}^{\infty} \mu(E_k) = \infty = \mu(E)$.

Case (ii): If $\{\|\mu_n(E \bigcap G_n)\|\}_{n=1}^{\infty}$ is bounded, then $\mu(E) = s\text{-}\lim_{n\to\infty} \mu_n(E \bigcap G_n) \in \mathcal{A}_+$. Let $\sum_{j=1}^{\infty} \mu(E_{k_j})$ be a subseries of $\sum_{k=1}^{\infty} \mu(E_k)$. Then, for $\rho \in \mathcal{A}_+^*$ and $m \in \mathbf{N}$, we have

$$0 \leqslant \sum_{j=1}^{m} \rho\mu(E_{k_j}) = \rho\mu(E_{k_1} \bigcup \cdots \bigcup E_{k_m}) \leqslant \rho\mu(E).$$

So $\sum_{j=1}^{\infty} \rho\mu(E_{k_j}) < \infty$. Since $\mathcal{A}_{sa}^* = \mathcal{A}_+^* - \mathcal{A}_+^*$, we conclude that $\sum_{j=1}^{\infty} \rho\mu(E_{k_j}) < \infty$ for all $\rho \in \mathcal{A}_{sa}^*$, that is, every subseries of $\sum_{k=1}^{\infty} \mu(E_k)$ is weakly convergent and hence, by a theorem of Orlicz and Pettis ([7], p.22), the series $\sum_{k=1}^{\infty} \mu(E_k)$ is norm convergent in $\mathcal{A}_+$. Now

$$\sum_{k=1}^{\infty} \mu(E_k) = \lim_{N\to\infty} \sum_{k=1}^{N} \mu(E_k) \leqslant \mu(E),$$

and for the given faithful normal state $\kappa \in \mathcal{A}_*$, we have

$$\kappa\Big(\sum_{k=1}^{\infty} \mu(E_k)\Big) = \sum_{k=1}^{\infty} \kappa\mu(E_k) = \kappa\mu(E).$$

Hence $\sum_{k=1}^{\infty} \mu(E_k) = \mu(E)$ by faithfulness of κ. ∎

REMARK. The above proof actually implies that

$$\mu(E) = \lim_{n\to\infty} \mu(E \bigcap G_n), \qquad E \in \mathcal{B}$$

where 'lim' denotes the norm limit if the sequence $\mu(E \bigcap G_n)$ is bounded, and is ∞ otherwise.

Let $\sigma : \mathcal{B} \to Z$ be as before and let $\mu \in M(G, \mathcal{A}_+)$, where $\mathcal{A}$ acts on a separable Hilbert space. Let $E \in \mathcal{B}$. We observe that the sets $\{y \in G : \mu(E-y) \leqslant r\mathbf{I}\}$ and $\{y \in G : r\mathbf{I} \leqslant \mu(E-y)\}$ are in $\mathcal{B}$ for $r \in \mathbf{R}$. Indeed, as $\mathcal{A}$ has separable predual, its normal states have a countable dense set $\{\rho_n\}$ and so

$$\{y \in G : \mu(E-y) \leqslant r\mathbf{I}\} = \bigcap_{n=1}^{\infty} \{y \in G : \rho_n\mu(E-y) \leqslant r\} \in \mathcal{B}.$$

Using Lemma 2.3 and the monotone closedness of $\mathcal{A}$, we can define the *convolution measure* $\mu * \sigma : \mathcal{B} \to \mathcal{A}_+ \bigcup \{\infty\}$ by

$$(\mu * \sigma)(E) = \begin{cases} \int_G \mu(E-y)\, \mathrm{d}\sigma(y) & \text{if the integral exists ;} \\ \infty & \text{otherwise} \end{cases}$$

where $E \in \mathcal{B}$.

Let $T(H)$ be the Banach space of trace-class operators on a Hilbert space H, equipped with the trace-norm $\|t\|_1 = \mathrm{tr}\,(|t|)$ so that the dual $T(H)^*$ identifies with $B(H)$ under the duality

$$(t, s) \in T(H) \times B(H) \longmapsto \mathrm{tr}\,(st) \in \mathbf{C}.$$

As a special case of Lemma 3.1 and 3.2, every positive linear functional φ on $K(G, T(H)_{sa})$ is continuous, and if H is separable, φ can be represented as a positive measure in $M(G, B(H)_+)$.

Let $K(H)$ be the C^*-algebra of compact operators on H (with the operator norm $\|\cdot\|$). Then $K(H)^*_{sa} = T(H)_{sa}$. If $\{t_n\}$ is an increasing sequence in $T(H)_+$ and if $\{\|t_n\|_1\}$ is bounded, then $\{\|t_n\|\}$ is bounded because $\|t\| \leqslant \|t\|_1$ for $t \in T(H)$. So $t = s - \lim_{n\to\infty} t_n$ exists in $B(H)_+$. But $0 \leqslant t_n \uparrow$ implies $\|t_n\|_1 = \mathrm{tr}(t_n)$ is increasing and therefore converges. It follows that, for $n \geqslant m$, we have

$$\|t_n - t_m\|_1 = \mathrm{tr}(t_n - t_m) = \mathrm{tr}(t_n) - \mathrm{tr}(t_m) \to 0$$

as $n, m \to \infty$. Hence $\{t_n\}$ is Cauchy in $T(H)$ and so $t = \lim_{n\to\infty} t_n \in T(H)_+$. Therefore $T(H)$ is *monotone closed* with respect to $\|\cdot\|_1$, and similar to Lemma 2.3, we have the identification

$$M(G, T(H)_+) = K(G, K(H)_{sa})^*_+$$

provided that H is separable so that $B(H)$ has a faithful normal state.

4. EXTREMAL SOLUTIONS

In view of the above discussions, we will only consider G a separable, metrizable, locally compact abelian group, H a separable Hilbert space and $\mathcal{A} \subset B(H)$ a von Neumann algebra with centre Z. For a fixed measure $\sigma : \mathcal{B} \to Z_+$ such that $\operatorname{supp} \sigma$ generates G, our objective is to solve the equation

$$\mu = \mu * \sigma$$

for $\mu \in M(G, \mathcal{A}_+)$. We are going to generalize Deny's method [7] to the above setting. We let

$$H_\sigma = \{\mu \in M(G, \mathcal{A}_+) : \mu = \mu * \sigma\}.$$

In this section, we characterize the extremal solutions in H_σ and we show in the next section that if $\mu \in H_\sigma$ is $T(H)_+$-valued as well, then it can be represented, via Choquet theory, by the extremal solutions in H_σ.

By Lemma 3.2, we identify $M(G, \mathcal{A}_+)$ with the cone $K(G, \mathcal{A}_*)^*_+$ of positive functionals in $K(G, \mathcal{A}_*)^*$. Clearly H_σ is a subcone of $M(G, \mathcal{A}_+)$.

Given a cone C in a real vector space and given a nonzero $u \in C$, let $R(u) = \{ru : r \geqslant 0\}$ be the *ray* in C generated by u. We call a nonzero u an *extremal element* in C if $R(u)$ is an *extreme ray* in C, that is, for any $v \in C, v \leqslant u$ implies $v \in R(u)$. Let ∂C denote the set of all extremal elements in C. Note that $0 \notin \partial C$.

We first describe the extremal elements of the cone $\mathcal{A}_+$ of any C^*-algebra $\mathcal{A}$. A nonzero projection $p \in \mathcal{A}$ is called *minimal* if $p\mathcal{A}p = \{\alpha p : \alpha \in \mathbb{C}\}$ (cf. [22], p.51).

LEMMA 4.1. *Let $p \in \mathcal{A}$ be a projection and let $b \in \mathcal{A}$ with $0 \leqslant b \leqslant p$. Then $b = bp = pb = pbp$. In particular, if p is a minimal projection, then $b = \alpha p$ for some $\alpha \geqslant 0$.*

Proof. We have

$$0 = (\mathbf{I} - p)0(\mathbf{I} - p) \leqslant (\mathbf{I} - p)b(\mathbf{I} - p) \leqslant (\mathbf{I} - p)p(\mathbf{I} - p) = 0$$

implying $(\mathbf{I} - p)b(\mathbf{I} - p) = 0$, that is, $((\mathbf{I} - p)b^{\frac{1}{2}})((\mathbf{I} - p)b^{\frac{1}{2}})^* = 0$ which gives $(\mathbf{I} - p)b^{\frac{1}{2}} = 0$ and hence $(\mathbf{I} - p)b = 0$. ∎

PROPOSITION 4.2. *The extremal elements of the cone $\mathcal{A}_+$ are precisely the positive scalar multiples of the minimal projections in $\mathcal{A}$.*

Proof. Let $t \in \mathcal{A}_+$ be extremal with $\|t\| = 1$, then t is an extreme point of the positive part of the unit ball $\{a \in \mathcal{A}_+ : \|a\| \leqslant 1\}$. Hence t is a projection in $\mathcal{A}$ (cf., [22], Lemma I.10.1). Now for any $a \in \mathcal{A}_+, 0 \leqslant tat \leqslant \|a\|t$ implies that $tat = \alpha t$ for some $\alpha \geqslant 0$, since t is extremal. It follows that $t\mathcal{A}t = \mathbb{C}t$ and t is a minimal projection in $\mathcal{A}$.

Conversely, if p is a minimal projection in $\mathcal{A}$ and if $t = \alpha p$ for some $\alpha > 0$, then for any $0 \leqslant b \leqslant t$, we have, by Lemma 4.1, $b = pbp = \beta p$ for some $\beta \geqslant 0$. So t is extremal in $\mathcal{A}_+$. ∎

We remark that the minimal projections in $B(H)$ are just the rank-one projections; in a von Neumann algebra minimal projection need not exist.

LEMMA 4.3. *Let $\mathcal{M}$ be a maximal abelian subalgebra of $\mathcal{A}$. If $p \in \mathcal{M}$ is a minimal projection in $\mathcal{M}$, then p is also minimal in $\mathcal{A}$.*

Proof. Let $a \in \mathcal{A}$. For any $b \in \mathcal{M}$, we have $bp = pb = \alpha p$ for some $\alpha \in \mathbb{C}$, so

$$(pap)b = pa(\alpha p) = \alpha pap = b(pap).$$

Hence $p\mathcal{A}p$ commutes with every element in $\mathcal{M}$ and so $p\mathcal{A}p \subseteq \mathcal{M}$ by maximality. Therefore $p\mathcal{A}p = \mathbb{C}p$ and p is minimal in $\mathcal{A}$. ∎

We now return to consider the extremal elements of H_σ.

LEMMA 4.4. *Let $\mu \in \partial H_\sigma$. Let $V \in \mathcal{B}$ and let σ_V be the restriction of σ to V. Then*

$$\mu * \sigma_V = \alpha\mu$$

for some $0 \leqslant \alpha \leqslant 1$.

Proof. For any compact set $K \subset G$ and for any pure state ρ of $\mathcal{A}$, we have

$$\begin{aligned}\rho\big((\mu * \sigma_V) * \sigma\big)(K) &= \big((\rho\mu * \rho\sigma_V) * \rho\sigma\big)(K) \\ &= \big((\rho\mu * \rho\sigma) * \rho\sigma_V\big)(K) \\ &= \rho\big((\mu * \sigma) * \sigma_V\big)(K).\end{aligned}$$

Therefore $(\mu * \sigma_V) * \sigma = \mu * \sigma_V$ and $\mu * \sigma_V$ is in H_σ. That μ is extremal and $\mu * \sigma_V \leqslant \mu * \sigma = \mu$ implies $\mu * \sigma_V = \alpha\mu$ for some $0 \leqslant \alpha \leqslant 1$. ∎

LEMMA 4.5. *Let $\mu \in \partial H_\sigma$ and let ρ be a state of $\mathcal{A}$ satisfying* (2.1) *(in particular, a pure state) and $\rho\mu \neq 0$. Then* $\operatorname{supp} \rho\sigma = \operatorname{supp} \sigma$.

Proof. Clearly $\operatorname{supp} \rho\sigma \subseteq \operatorname{supp} \sigma$. To prove the reverse inclusion, let $x \in \operatorname{supp} \sigma$, let V be any compact neighbourhood of x, and let σ_V be the restriction of σ to V. Then $\mu * \sigma_V \neq 0$ (indeed $\operatorname{supp} \mu * \sigma_V = \operatorname{supp} \mu + \operatorname{supp} \sigma_V$). If $\rho\sigma(V) = 0$, then $\rho\sigma_V = 0$ and by Lemma 4.4 and (2.1), we have

$$\alpha(\rho\mu) = \rho\mu * \rho\sigma_V = 0.$$

This contradicts the hypothesis that $\rho\mu \neq 0$, so $\rho\sigma(V) \neq 0$ and $x \in \operatorname{supp} \rho\sigma$. ∎

We note that $\rho\mu = 0$ can occur. Indeed, let $\mathcal{A} = M_2(\mathbf{C})$ and define $\rho \in \mathcal{A}^*$ by

$$\rho \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \operatorname{tr} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then ρ is a pure state of $\mathcal{A}$ ([21], Lemma 8.3), and if $\mu : \mathcal{B} \to \mathcal{A}_+$ is given by

$$\mu(\cdot) = \begin{bmatrix} \nu(\cdot) & \nu(\cdot) \\ \nu(\cdot) & \nu(\cdot) \end{bmatrix}$$

where $\nu : \mathcal{B} \to [0, \infty)$ is any scalar measure, then $\rho\mu = 0$.

We note that the centre Z is a commutative von Neumann algebra and can be identified with the algebra $C(\Omega)$ of complex continuous functions on the pure state space

$$\Omega = \{\omega \in Z^* : \omega \text{ is a pure state of } Z\},$$

which is w*-compact Hausdorff and Stonean ([22], p.104). There is a positive contractive projection $P : \mathcal{A} \to Z$ such that

$$P(az) = P(a)z \quad \text{for} \quad a \in \mathcal{A} \quad \text{and} \quad z \in Z$$

and that P is *faithful*, i.e., $P(a) = 0$ and $a \geqslant 0 \Rightarrow a = 0$, [23]. It follows that $\tilde{\omega} := \omega P$ is a state of $\mathcal{A}$ satisfying (2.1) for $\omega \in \Omega$. By faithfulness of P, the set

$$U = \{\omega \in \Omega : \tilde{\omega}\mu \neq 0\}$$

is nonempty.

LEMMA 4.6. *Let $\mu \in \partial H_\sigma$ and let $K \in \mathcal{B}$ be such that $\mu(K) \neq 0$. Then $P\mu(K)$ is an extremal element in the cone Z_+.*

Proof. By faithfulness of P, $P\mu(K) \neq 0$. Let $b \in Z_+ \setminus \{0\}$ be such that $b \leqslant P\mu(K)$. We show that b is a positive scalar multiple of $P\mu(K)$. Let

$$b_n = \left(b + \frac{1}{n}\mathbf{I}\right)\left(P\mu(K) + \frac{1}{n}\mathbf{I}\right)^{-1} \in Z_+,$$

then $b + \frac{1}{n}(\mathbf{I} - b_n) = b_n P\mu(K)$, and $0 \leqslant b_n \leqslant \mathbf{I}$. We define a measure $\nu : \mathcal{B} \to \mathcal{A}_+ \bigcup\{\infty\}$ by $\nu(E) = b_n\mu(E)$ if $\mu(E) \in \mathcal{A}$, and $= \infty$ otherwise. Since b_n commutes with $\mu(E)$, we have $\nu \leqslant \mu$. Evidently $\nu \in H_\sigma$, and therefore $\nu = c_n\mu$ for some $c_n > 0$ since $\mu \in \partial H_\sigma$. This implies

$$b + \frac{1}{n}(\mathbf{I} - b_n) = b_n P\mu(K) = P(b_n\mu(K)) = P(\nu(K)) = P(c_n\mu(K)) = c_n P\mu(K).$$

Since $b = \lim_{n\to\infty}\left(b + \frac{1}{n}(\mathbf{I} - b_n)\right)$, it follows that $\{c_n\}$ converges to $c > 0$ say, and $b = cP\mu(K)$. So $P\mu(K)$ is extremal in Z_+. ∎

LEMMA 4.7. *Let $\mu \in \partial H_\sigma$. Then there is a minimal projection q in Z such that $P\mu(G_n) = e_n q$ for some $e_n \geqslant 0$.*

Proof. By Lemma 4.6 and Proposition 4.2, we have $P\mu(G_n) = e_n q_n$ for some $e_n \geqslant 0$ and some minimal projection q_n in Z. But given $n \leqslant m$ say, $e_n q_n = P\mu(G_n) \leqslant P\mu(G_m) = e_m q_m$ implies $q_n = q_m$ by Lemma 4.1. ∎

LEMMA 4.8. *Let $\mu \in \partial H_\sigma$ and let ρ be a pure state of $\mathcal{A}$ with $\rho\mu \neq 0$. Then $\rho P\mu \neq 0$.*

Proof. We have $\rho\mu(G_n) \neq 0$ for some n which implies $P\mu(G_n) = e_n q$ with $e_n > 0$ by Lemma 4.7. Since q commutes with $\mu(G_n)$, we have $q\mu(G_n) \leqslant \mu(G_n)$. Now

$$P(q\mu(G_n)) = qP(\mu(G_n)) = e_n q = P(\mu(G_n))$$

entails $q\mu(G_n) = \mu(G_n)$ by faithfulness of P. So $\rho P\mu(G_n) = e_n\rho(q) > 0$. ∎

We note that if p is a projection in Z and if ρ is a pure state of $\mathcal{A}$, then $\rho(p) = 1$ or 0.

LEMMA 4.9. *Let* $\mu \in \partial H_\sigma$ *and let* $x \in G$. *Then*

$$\mu * \delta_x = g(x)\mu$$

where δ_x *is the point mass at* x *and* $g : G \to (0,\infty)$ *satisfies* $g(x+y) = g(x)g(y)$.

Proof. Fix $x \in \operatorname{supp}\sigma$. Let $\{V_n\}$ be a decreasing sequence of compact neighbourhoods of x such that $\lim_{n\to\infty} V_n = \{x\}$. Let $\sigma_n = \sigma|V_n$ be the restriction of σ to V_n. By Lemma 4.4, we have

$$\mu * \sigma_n = \alpha_n \mu$$

for some $0 < \alpha_n < 1$. Let ρ be a state of $\mathcal{A}$ satisfying (2.1) such that $\rho\mu \neq 0$. By Lemma 4.5, we have $\operatorname{supp}\rho\sigma = \operatorname{supp}\sigma$ and so $\rho\sigma(V_n) \neq 0$. Now we can apply Deny's arguments [7], making use of the sequence $\{\frac{1}{\rho\sigma(V_n)}\rho\sigma_n\}$ which converges vaguely to δ_x, to conclude that

$$\rho\mu * \delta_x = g_\rho(x)\rho\mu \tag{4.1}$$

where $g_\rho(x) = \lim_{n\to\infty} \frac{\alpha_n}{\rho\sigma(V_n)}$ satisfies

$$g_\rho(x+y) = g_\rho(x)g_\rho(y) \tag{4.2}$$

for $y,\ x+y \in \operatorname{supp}\sigma$. Further, since $\operatorname{supp}\sigma$ generates G, we can extend g_ρ to a continuous function $g_\rho : G \to (0,\infty)$ such that (4.1) and (4.2) hold for all $x, y \in G$. We remark that if ρ and ρ' are two states satisfying (2.1) with the same restriction to Z, then $g_\rho = g_{\rho'}$. Note that $g_\rho(0) = 1$.

To construct the required g, we make use of the aforementioned projection $P : \mathcal{A} \to Z$. For each $\omega \in U$, we have

$$\tilde{\omega}\mu * \delta_x = g_{\tilde{\omega}}(x)\tilde{\omega}\mu, \qquad x \in G.$$

We show that $g_{\tilde{\omega}_1}(x) = g_{\tilde{\omega}_2}(x)$ for all $\omega_1, \omega_2 \in U$. Indeed, there is some G_m such that both $\tilde{\omega}_1\mu(G_m)$ and $\tilde{\omega}_2\mu(G_m)$ are positive which implies

$$\tilde{\omega}_1\mu(G_m) = \omega_1 P\mu(G_m) = \omega_1(e_m q) = e_m = \tilde{\omega}_2\mu(G_m) > 0.$$

So $\omega_1 P\mu(G_m - x) = g_{\tilde{\omega}_1}(x)\omega_1 P\mu(G_m) > 0$ entails that $P\mu(G_m - x) = e'_m q'$ for some positive e'_m and some minimal projection $q' \in Z$. It follows that

$$g_{\tilde{\omega}_1}(x) = \frac{e'_m}{e_m} = g_{\tilde{\omega}_2}(x).$$

Fix any $\omega_0 \in U$, we define $g : G \to (0,\infty)$ by $g(x) = g_{\tilde{\omega}_0}(x)$ for $x \in G$, then $g(x+y) = g(x)g(y)$ for $x, y \in G$.

Now for each pure state ρ of $\mathcal{A}$ with $\rho\mu \neq 0$ and $\omega = \rho|Z$, we have $\omega \in U$ by Lemma 4.8. Using (4.1), we have

$$\rho(\mu * \delta_x) = \rho\mu * \delta_x = g_\rho(x)\rho\mu = g_{\tilde{\omega}}(x)\rho\mu = g_{\tilde{\omega}_0}(x)\rho\mu = g(x)\rho\mu.$$

Hence

$$\mu * \delta_x = g(x)\mu. \quad \blacksquare$$

LEMMA 4.10. *Let $\mu \in \partial H_\sigma$. Then*

$$\mathrm{d}\mu(x) = ag(-x)\mathrm{d}\lambda(x)$$

where $a \in \mathcal{A}$, λ is the Haar measure on G, and $g : G \to (0,\infty)$ satisfies $g(x+y) = g(x)g(y)$.

Proof. For each n, define ν_n by

$$\nu_n(E) = \int\limits_{E\cap G_n} g(x)\,\mathrm{d}\mu(x), \qquad E \in \mathcal{B}.$$

By the remark following Lemma 2.3, the vector integral is defined by the bilinear map $(z,a) \in Z \times \mathcal{A} \mapsto za \in \mathcal{A}$. Also we define

$$\nu(E) = \lim_{n\to\infty} \nu_n(G_n \textstyle\bigcap E), \quad E \in \mathcal{B},$$

where 'lim' is the norm limit (see Lemma 3.2 and the remark there). It follows that for any pure state ρ of $\mathcal{A}$, $\mathrm{d}\rho\nu(x) = g(x)\mathrm{d}\rho\mu(x)$. We denote ν by

$$\mathrm{d}\nu(x) = g(x)\mathrm{d}\mu(x).$$

Using (4.2), it is elementary to show that $\rho\nu$ is actually translation invariant [7]. Since pure states separate points of $\mathcal{A}$, ν is translation invariant as well. We conclude that for *any* state ρ of $\mathcal{A}$, $\rho\nu$ is a (scalar) translation invariant regular Borel measure, hence there exists $a(\rho) \in [0,\infty)$ such that

$$\rho\nu = a(\rho)\lambda \tag{4.3}$$

where λ is the Haar measure on G.

It is easy to see that the function $a(\cdot)$ is affine on the state space of $\mathcal{A}$. It is also continuous with respective to the $\sigma(\mathcal{A}^*,\mathcal{A})$-topology. Indeed let $\{\rho_\alpha\}$ be a net of states of $\mathcal{A}$, $\sigma(\mathcal{A}^*,\mathcal{A})$-converging to ρ. Let $K \subset G$ such that $\lambda(K) \neq 0$, then $\rho_\alpha\nu(K) \to \rho\nu(K)$ and (4.3) implies that $a(\rho_\alpha) \to a(\rho)$. Therefore $a(\rho)$ is a nonnegative continuous affine function of the states ρ of $\mathcal{A}$ and it defines a positive operator, denoted by a, in $\mathcal{A}$ (cf., [22], p.161). Hence we have $\nu(K) = a\lambda(K)$ for every compact set $K \subset G$. For every pure state ρ of $\mathcal{A}$, we have

$$\mathrm{d}\rho\mu = g(-x)\mathrm{d}\rho\nu = g(-x)a(\rho)\mathrm{d}\lambda$$

and hence

$$\mathrm{d}\mu(x) = ag(-x)\mathrm{d}\lambda(x). \quad \blacksquare$$

We are now ready to characterize the extremal solutions of the equation $\mu * \sigma = \mu$. Recall that $\mathcal{A}$ acts on a separable Hilbert space.

THEOREM 4.11. *Let $\mu \in H_\sigma$. The following conditions are equivalent:*
(i) $\mu \in \partial H_\sigma$;
(ii) $\mathrm{d}\mu(x) = cpg(x)\mathrm{d}\lambda(x)$ *where* $c > 0$, *λ is the Haar measure on G, p is a minimal projection in $\mathcal{A}$, and the function $g : G \to [0, \infty)$ has the properties that*

$$g(x+y) = g(x)g(y) \quad \text{and} \quad p = p\left(\int_G g(-y)\,\mathrm{d}\sigma(y)\right).$$

Proof. (i) $\Rightarrow$ (ii). By the previous lemma, we have

$$\mathrm{d}\mu(x) = ag(-x)\,\mathrm{d}\lambda(x)$$

where $a \in \mathcal{A}_+$ and $g : G \to (0, \infty)$ satisfies $g(x+y) = g(x)g(y)$. We show that a is extremal in $\mathcal{A}_+$. We first note that μ has commuting range in $\mathcal{A}_+$ which means, for $\mu(E), \mu(F) \in \mathcal{A}_+$, $\mu(E)\mu(F) = \mu(F)\mu(E)$. Hence there is a maximal abelian subalgebra $\mathcal{M} \subset \mathcal{A}$ such that $\mu : \mathcal{B} \to \mathcal{M}_+ \bigcup\{\infty\}$. Let $K \subset G$ be a compact set such that $\mu(K) \neq 0$. We have

$$\mu(K) = a \int_K g(-x)\,\mathrm{d}\lambda(x).$$

Using similar arguments as in Lemma 4.6, one can show that $\mu(K)$ is extremal in $\mathcal{M}_+$ and hence $a \in \partial\mathcal{M}_+$. By Proposition 4.2 and Lemma 4.3 there exists a minimal projection $p \in \mathcal{A}$ such that $a = cp$ for some $c > 0$. Hence we have $\mathrm{d}\mu(x) = cpg(-x)\mathrm{d}\lambda(x)$.

It remains to prove the last identity in (ii). Since $\mu = \mu * \sigma$, and $\mathrm{d}\mu(x) = cpg(-x)\mathrm{d}\lambda(x)$ we have, by a direct calculation,

$$\int_K pg(-x)\,\mathrm{d}\lambda(x) = \int_G\int_K pg(-x)g(-y)\,\mathrm{d}\lambda(x)\,\mathrm{d}\sigma(y),$$

for any compact $K \subset G$. It follows from (2.1) that for any pure state ρ of $\mathcal{A}$,

$$\begin{aligned}\rho(p)\Big(\int_K g(-x)\,\mathrm{d}\lambda(x)\Big) &= \rho\Big(p\int_G\int_K g(-x)g(-y)\,\mathrm{d}\lambda(x)\,\mathrm{d}\sigma(y)\Big)\\ &= \rho(p)\Big(\int_K g(-x)\,\mathrm{d}\lambda(x)\Big)\Big(\int_G g(-y)\,\mathrm{d}\rho\sigma(y)\Big)\\ &= \Big(\int_K g(-x)\,\mathrm{d}\lambda(x)\Big)\rho\Big(p\int_G g(-y)\,\mathrm{d}\sigma(y)\Big).\end{aligned}$$

We conclude that $p = p\int_G g(-y)\,\mathrm{d}\sigma(y)$.

(ii) $\Rightarrow$ (i). Let μ satisfy condition (ii) and let $\nu \in H_\sigma$ be such that $\nu \leqslant \mu$. We show that ν is a positive scalar multiple of μ. Let $\mathrm{d}\mu(x) = cpg(x)\,\mathrm{d}\lambda(x)$ be as given. Define $\hat{\nu} \in M(G, \mathcal{A}_+)$ and $\hat{\sigma} : \mathcal{B} \to \mathcal{A}_+$ by

$$\mathrm{d}\hat{\nu}(x) = g(-x)\mathrm{d}\nu(x) \quad \text{and} \quad \mathrm{d}\hat{\sigma}(x) = g(-x)\mathrm{d}\sigma(x).$$

It follows from a direct calculation that, for any compact subset $K \subset G$,

$$(\hat{\nu} * \hat{\sigma})(K) = \hat{\nu}(K),$$

so that $\hat{\nu} \in H_{\hat{\sigma}}$. Let $\hat{\nu}_x = \hat{\nu} * \delta_x$ be a translation of ν and let h be a positive real continuous function with compact support on G. Define $f : G \to \mathcal{A}_{sa}$ by $f(x) = \int_G h(y)\,\mathrm{d}\hat{\nu}_{-x}(y)$, then

$$\int_G f(x-y)\,\mathrm{d}\hat{\sigma}(y) = f(x)$$

for all $x \in G$. Let ρ be a pure state of $\mathcal{A}$ such that $\rho\nu \neq 0$. Then $\rho\mu \neq 0$ and $\rho(p) \neq 0$. Also

$$\int_G \rho f(x-y)\,\mathrm{d}\rho\hat{\sigma}(y) = \rho f(x)$$

where $\rho\hat{\sigma}(G) = 1$ as $p\hat{\sigma}(G) = p$ by the last identity in (ii). Since $\rho f(x) = \int_G h(y)\,\mathrm{d}\rho\hat{\nu}_{-x}(y)$ with $\rho\hat{\nu} \leqslant c\rho(p)\lambda$, the function $\rho f : G \to \mathbf{R}$ is bounded and uniformly continuous. Therefore by Choquet and Deny's Theorem ([5], Théorème 1), we have

$$\rho f(x-a) = \rho f(x)$$

for $x \in G$ and $a \in \operatorname{supp} \rho\hat{\sigma}$. By Lemma 4.5, $\operatorname{supp} \rho\hat{\sigma} = \operatorname{supp} \rho\sigma = \operatorname{supp} \sigma$. Since $\operatorname{supp} \sigma$ generates the group G, we conclude that $\rho f(x-a) = \rho f(x)$ for all $x, a \in G$. In particular, $\rho f(-a) = \rho f(0)$ for $a \in G$, that is

$$\int_G h(y)\, \mathrm{d}\rho\hat{\nu}_a(y) = \int_G h(y)\, \mathrm{d}\rho\hat{\nu}(y).$$

As h is arbitrary, we have $\rho\hat{\nu}_a = \rho\hat{\nu}$ for $a \in G$ and hence $\rho\hat{\nu} = a(\rho)\lambda$ for some $a(\rho) \in (0, \infty)$, where λ is the Haar measure on G.

That $\nu \leqslant \mu$ implies $0 \leqslant \hat{\nu}(K) \leqslant cp\lambda(K)$ where p is a minimal projection in $\mathcal{A}_+$. By Proposition 4.2, there exists $\alpha_K \in (0, \infty)$ such that $\hat{\nu}(K) = \alpha_K cp\lambda(K)$. It follows that, if $\lambda(K) \neq 0$, then $\rho\hat{\nu} = a(\rho)\lambda$ gives $a(\rho) = \alpha_K c$. This shows that α_K does not depend on K and so we have $\hat{\nu} = \alpha cp\lambda$ for some $\alpha \in (0, \infty)$. Hence

$$\mathrm{d}\nu(x) = g(x)\mathrm{d}\hat{\nu}(x) = \alpha cpg(x)\mathrm{d}\lambda(x) = \alpha \mathrm{d}\mu(x).$$

Therefore $\mu \in \partial H_\sigma$. The proof is complete. ∎

5. GENERAL SOLUTIONS

We have seen in Theorem 4.11 that the existence of extremal solutions $\mu : \mathcal{B} \to \mathcal{A}_+ \bigcup\{\infty\}$ for the equation $\mu = \mu * \sigma$ depends on the existence of minimal projections in $\mathcal{A}$. Therefore we have to restrict ourselves to the class of von Neumann algebras rich in minimal projections. These are the so-called *atomic* von Neumann algebras. Recall that a von Neumann algebra $\mathcal{A}$ is called *atomic* if every nonzero projection in $\mathcal{A}$ majorizes a nonzero minimal projection ([22], p.155). A typical example of an atomic von Neumann algebra is $B(H) \otimes \ell^\infty$ in which ℓ^∞ is the centre.

Henceforth $\mathcal{A}$ will denote an *atomic* von Neumann algebra acting on a suitably chosen *separable* Hilbert space H so that there is a positive contractive projection $\mathsf{E} : B(H) \to \mathcal{A}$ with the following properties:

(i) $\mathsf{E}(atb) = a\mathsf{E}(t)b$ for $a, b \in \mathcal{A}$ and $t \in B(H)$;

(ii) E continuous with respect to the w*-topologies on $B(H)$ and $\mathcal{A}$;

(iii) $\mathbf{tr} \circ \mathsf{E} = \mathbf{tr}$ where $\mathbf{tr}$ denotes the canonical trace on $B(H)$.

The projection E is called a *conditional expectation* and its existence has been shown, for instance, in ([22], p.334 and Proposition V.2.36). Note that in the above representation of $\mathcal{A}$, the minimal projections in $\mathcal{A}$ are rank-one projections on H. By (ii), there exists a map

$$\mathsf{E}_* : \mathcal{A}_* \to T(H)_{sa}$$

induced by E on the preduals (recall $\mathcal{A}_*$ is the predual of $\mathcal{A}_{sa}$) by transpose: $\mathsf{E}_*(\rho) = \rho \circ \mathsf{E}$. Since $\mathcal{A}$ has a separable predual, there is a countable set of normal states separating points of $\mathcal{A}$. Further, the atomicity of $\mathcal{A}$ implies that its normal state space is the norm-closed convex hull of the pure normal states and therefore there is a countable set $\{\rho_n\}$ of *pure* normal states separating points of $\mathcal{A}$. In particular, given $\mu, \nu \in M(G, \mathcal{A}_+)$ with $\rho_n\mu \leqslant \rho_n\nu$ for all n, then $\mu \leqslant \nu$. In the sequel, $\{\rho_1, \rho_2, \ldots, \rho_n, \ldots\}$ will always denote the above set of pure normal states of $\mathcal{A}$.

In order to use Choquet's representation theory, we first note that the cone H_σ need not be closed in $M(G, \mathcal{A}_+)$, and therefore we need to introduce the following auxiliary cone as in [7]:

$$C_\sigma = \{\mu \in M(G, \mathcal{A}_+) : \mu * \sigma \leqslant \mu\}.$$

LEMMA 5.1. *The cone C_σ is a w*-closed subcone of $M(G, \mathcal{A}_+)$.*

Proof. Let $\{\mu_\alpha\}$ be a net in C_σ w*-converging to $\mu \in M(G, \mathcal{A}_+)$. We observe that for any $h : G \to \mathbf{R}_+$ continuous with compact support and for any pure normal state ρ of $\mathcal{A}$,

$$\rho\mu_\alpha(h) = \mu_\alpha(h(\cdot)\rho) \to \mu(h(\cdot)\rho) = \rho\mu(h)$$

where $h(\cdot)\rho \in K(G, \mathcal{A}_*)$. By $\mu_\alpha * \sigma \leqslant \mu_\alpha$ and by the Fatou Lemma, we have

$$\rho(\mu * \sigma)(h) = (\rho\mu * \rho\sigma)(h) \leqslant \rho\mu(h).$$

Since h is arbitrary, we have $\rho\mu * \sigma \leqslant \rho\mu$. Also since the pure normal states separates points of $\mathcal{A}$, we conclude that $\mu * \sigma \leqslant \mu$ and $\mu \in C_\sigma$. ∎

LEMMA 5.2. *Let $\mu \in H_\sigma$. Then μ is extremal in H_σ if and only if μ is extremal in C_σ; that is, $\partial H_\sigma = \partial C_\sigma \bigcap H_\sigma$.*

Proof. Let $\mu \in \partial H_\sigma$ and let $\nu \in C_\sigma$ be such that $\mu - \nu \in C_\sigma$. Then

$$\mu = \mu * \sigma = (\mu - \nu) * \sigma + \nu * \sigma \leqslant (\mu - \nu) + \nu = \mu,$$

which implies $\nu * \sigma = \nu$, that is, $\nu \in H_\sigma$ and hence $\nu = c\mu$ for some $c \geqslant 0$. This shows that $\mu \in \partial C_\sigma$. ∎

Let C be a closed cone in a locally convex space. By a *cap* of C we mean a compact convex subset $\mathcal{K}$ of C containing 0 and is such that $C \setminus \mathcal{K}$ is convex; C is called *well-capped* if C is a union of caps ([4], p.202).

LEMMA 5.3. *Let C be a w^*-closed subcone of $M(G,\mathcal{A}_+)$. Then C is w^*-complete and every cap of C is w^*-metrizable.*

Proof. We first show that $M(G,\mathcal{A}_+)$ is w*-complete, so that C will be w*-complete as well. Let $\{\mu_\alpha\}$ be a w*-Cauchy net in $M(G,\mathcal{A}_+)$. Then $\{\mu_\alpha(f)\}$ is Cauchy in $\mathbf{R}$ for every $f \in K(G,\mathcal{A}_*)$ and converges to $\mu(f)$ say, which defines a positive linear functional μ on $K(G,\mathcal{A}_*)$. By Lemma 3.2, $\mu \in K(G,\mathcal{A}_*)^*_+ = M(G,\mathcal{A}_+)$ and the assertion follows.

Note that $K(G,\mathbf{R})^*_+$ is w*-complete and metrizable ([3], Theorem 12.2 and Theorem 12.10). Let $\{\rho_n\}$ be the pure normal states as described before Lemma 5.1, consider the mapping

$$\mu \in M(G,\mathcal{A}_+) \mapsto (\rho_1\mu,\ldots,\rho_n\mu,\ldots) \in \prod_{n\in\mathbf{N}} C_n$$

where $C_n = K(G,\mathbf{R})^*_+$, and $\prod_{n\in\mathbf{N}} C_n$, equipped with the product topology, is complete and metrizable. The map is one-to-one and continuous, therefore, given a cap $\mathcal{K} \subset C$, the restriction

$$\mu \in \mathcal{K} \mapsto (\rho_1\mu,\ldots,\rho_n\mu,\ldots) \in \prod_{n\in\mathbf{N}} C_n$$

is a homeomorphic embedding, by compactness of $\mathcal{K}$. Hence $\mathcal{K}$ is metrizable. ∎

Now we have shown that C_σ is w*-complete, one may attempt at this stage to use Choquet's theory for weakly complete cones, as in [7], to show that every solution $\mu \in H_\sigma \subset C_\sigma$ can be represented by a probability measure supported by the extreme rays ∂H_σ which have been characterized in Theorem 4.11. We encounter an obstacle here as it is not clear to us if C_σ is well-capped. On the other hand, we observe that, by Theorem 4.11, each $\mu \in \partial H_\sigma$ is in fact an extended $T(H)_+$-valued (i.e., $T(H)_+ \bigcup\{\infty\}$) measure and therefore, one expects that measures representable by ∂H_σ to be $T(H)$-valued as well. This suggests that we should consider the $T(H)_+$-valued measures in C_σ, and indeed, such a measure is contained in a cap of C_σ for which one can apply Choquet's theory.

Let $\mathbf{E} : B(H) \to \mathcal{A}$ be the aforementioned conditional projection and let $\mathbf{E}_* : \mathcal{A}_* \to T(H)_{sa}$ be the transpose of $\mathbf{E}$. We define another induced map $\tilde{\mathbf{E}} : M(G,T(H)_+) \to M(G,\mathcal{A}_+)$ by

$$\tilde{\mathbf{E}}\mu(S) = \begin{cases} \mathbf{E}(\mu(S)) & \text{if } \mu(S) \in T(H)_+ \\ \infty & \text{otherwise.} \end{cases}$$

Then $\tilde{\mathbf{E}}$ is w*-w*-continuous also. Indeed, let $\{\mu_\alpha\}$ be w*-convergent to μ in $M(G,T(H)_+)$ and let $h \in K(G,\mathcal{A}_*)$. Then $\mathbf{E}_*(h(\cdot)) \in K(G,T(H)_{sa})$. Since

$T(H) \subset K(H)$ and since the trace-norm dominates the operator-norm, we have $K(G, T(H)_{sa}) \subset K(G, K(H)_{sa})$ and so

$$(\tilde{\mathsf{E}}\mu_\alpha)(h) = \mu_\alpha(\mathsf{E}_*(h(\cdot))) \to \mu(\mathsf{E}_*(h(\cdot))) = (\tilde{\mathsf{E}}\mu)(h).$$

LEMMA 5.4. *Let $\tilde{\mathsf{E}}$ be defined as above.*
(i) *For $\mu \in M(G, \mathcal{A}_+) \bigcap M(G, T(H)_+)$, we have $\mu = \tilde{\mathsf{E}}\mu$;*
(ii) *$M(G, \mathcal{A}_+) \bigcap M(G, T(H)_+) = \{\tilde{\mathsf{E}}\mu : \mu \in M(G, T(H)_+)\}$.*

Proof. (i) is clear. For (ii) we need only observe that for $\mu \in M(G, T(H)_+)$, then $\tilde{\mathsf{E}}\mu \in M(G, T(H)_+)$ also since

$$\mu(S) \in T(H)_+ \;\Rightarrow\; \mathbf{tr}\,(\tilde{\mathsf{E}}\mu(S)) = (\mathbf{tr} \circ \mathsf{E})(\mu(S)) = \mathbf{tr}\,(\mu(S)) < \infty. \quad \blacksquare$$

PROPOSITION 5.5. *Let $\mu \in M(G, \mathcal{A}_+) \bigcap M(G, T(H)_+)$. Then there is a cap $\mathcal{K} \subset M(G, \mathcal{A}_+)$ such that $\mu \in \mathcal{K}$.*

Proof. Define a mapping $\nu \in M(G, T(H)_+) \mapsto (\nu_1, \ldots, \nu_n, \ldots) \in \prod_{n=1}^{\infty} M(G_n, T(H)_+)$ where ν_n is the restriction of ν to G_n. Let $\alpha_n > 0$ be such that $\alpha_n |||\mu_n||| \leqslant 1$ where $|||\cdot|||$ denotes the variation norm of a $T(H)$-valued measure on G_n with the trace norm on $T(H)$, i.e., $|||\mu_n||| = \|\mu_n(G_n)\|_1$. We first show that the convex set

$$B = \Big\{\nu \in M(G, T(H)_+) : \sum_{n=1}^{\infty} \frac{\alpha_n |||\nu_n|||}{2^n} \leqslant 1\Big\}$$

is a cap in $M(G, T(H)_+)$. Indeed B is homeomorphic, via the above mapping, with a closed subset in

$$\prod_{n=1}^{\infty} \Big\{\nu \in M(G_n, T(H)_+) : |||\nu||| \leqslant \frac{2^n}{\alpha_n}\Big\}$$

which is compact in the product topology. Also, $M(G, T(H)_+) \setminus B$ is convex since for $\nu, \tau \notin B$ and for $0 < r < 1$, we have by additivity of the norm

$$\sum_{n=1}^{\infty} \frac{\alpha_n |||r\nu_n + (1-r)\tau_n|||}{2^n} = r \sum_{n=1}^{\infty} \frac{\alpha_n |||\nu_n|||}{2^n} + (1-r) \sum_{n=1}^{\infty} \frac{\alpha_n |||\tau_n|||}{2^n} > 1$$

which implies $r\nu + (1-r)\tau \notin B$.

Now let $\mathcal{K} = \{\tilde{\mathsf{E}}\nu : \nu \in B\} \subset M(G, \mathcal{A}_+)$. Then $\mathcal{K}$ is compact in $M(G, \mathcal{A}_+)$ since we have shown that $\tilde{\mathsf{E}}$ is w*-w*-continuous. Evidently $\mathcal{K}$ is convex. We show that $M(G, \mathcal{A}_+) \setminus \mathcal{K}$ is convex. Let $\tau, \gamma \in M(G, \mathcal{A}_+) \setminus \mathcal{K}$. Suppose $\nu = \frac{1}{2}\tau + \frac{1}{2}\gamma \in \mathcal{K}$, we deduce a contradiction. Note that $2\nu \geqslant \tau, \gamma$ implies $\tau, \gamma \in M(G, T(H)_+)$ and it follows that $\tau, \gamma \notin B$ (otherwise $\tau = \tilde{\mathbf{E}}\tau \in \mathcal{K}$ and $\gamma = \tilde{\mathbf{E}}\gamma \in \mathcal{K}$). Suppose $\nu = \tilde{\mathbf{E}}\nu'$ for some $\nu' \in B$. Then

$$|||\frac{1}{2}\tau_n + \frac{1}{2}\gamma_n||| = |||\nu_n||| = |||(\tilde{\mathbf{E}}\nu')_n||| = |||\tilde{\mathbf{E}}\nu'_n||| = \mathbf{tr}\Big(\tilde{\mathbf{E}}\nu'_n(G_n)\Big)$$
$$= \mathbf{tr}\,(\nu'_n(G_n)) = |||\nu'_n|||$$

and therefore

$$1 < \frac{1}{2}\sum_{n=1}^{\infty}\frac{\alpha_n|||\tau_n|||}{2^n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\alpha_n|||\gamma_n|||}{2^n} = \sum_{n=1}^{\infty}\frac{\alpha_n|||\frac{1}{2}\tau_n + \frac{1}{2}\gamma_n|||}{2^n} = \sum_{n=1}^{\infty}\frac{\alpha_n|||\nu'_n|||}{2^n} \leqslant 1$$

giving a contradiction. So $\mathcal{K}$ is a cap in $M(G, \mathcal{A}_+)$ containing μ. ∎

Now we are in a position to apply Choquet's theory to describe the $T(H)_+$-valued measures in C_σ, and in particular, such measures in H_σ. We refer to ([4], Section 30) for the theory of conical measures on weakly complete cones.

THEOREM 5.6. *Let $\mathcal{A}$ be an atomic von Neumann algebra acting on a separable Hilbert space H, with centre Z. Let σ be a positive Z-valued measure on G. Given $\mu \in M(G, \mathcal{A}_+) \bigcap M(G, T(H)_+)$ and $\mu = \mu * \sigma$, then there is a probability measure $\mathbf{P}$ on H_σ supported by a Borel subset B of $\partial H_\sigma \bigcup \{0\}$ such that*

$$\mu = \int_B \nu \, \mathrm{d}\mathbf{P}(\nu)$$

where the integral means $\mu(h) = \int_B \nu(h)\,\mathrm{d}\mathbf{P}(\nu)$ for all $h \in K(G, \mathcal{A}_*)$.

Proof. Recall from Lemma 5.3 that C_σ is w*-complete and that every cap $\mathcal{K}$ of C_σ is w*- metrizable, and hence the set $\partial_e\mathcal{K}$ of extreme points of $\mathcal{K}$ is a w*-G_δ-set ([17], p.7). We also note that the rays generated by the elements of $\partial_e\mathcal{K}$ is contained in $\partial C_\sigma \bigcup \{0\}$ ([4], Proposition 30.12). By Proposition 5.5, every $\mu \in C_\sigma \bigcap M(G, T(H)_+)$ is contained in a cap of C_σ. A direct application of Choquet's integral representation theory yields (cf. [4], Theorem 30.14, Theorem 30.22)

$$\mu = c \int_{\partial_e\mathcal{K}} \nu \, \mathrm{d}\mathbf{P}(\nu) \tag{5.1}$$

where $c \geqslant 0$, P is a probability measure supported by $\partial_e\mathcal{K}$, and the integral means $\mu(h) = \int_{\partial_e\mathcal{K}} \nu(h)\,\mathrm{d}\mathbf{P}(\nu)$ for every $h \in K(G, \mathcal{A}_*)$.

To replace the set $\partial_e \mathcal{K}$ by a subset B of $\partial H_\sigma \bigcup \{0\}$ in the above integral representation, we first show that H_σ is a Borel set. Observe that

$$H_\sigma = \bigcap_{n=1}^{\infty} \{\nu \in M(G, \mathcal{A}_+) : \rho_n \nu = \rho_n \nu * \rho_n \sigma\}.$$

One can show, as in ([18], Lemma 9.5.2), that the set

$$\{\tau \in M(G, \mathbf{R}_+) : \tau * \rho_n \sigma = \tau\}$$

is a w*-Borel set in $M(G, \mathbf{R}_+)$. As the map

$$\nu \in M(G, \mathcal{A}_+) \mapsto \rho_n \nu \in M(G, \mathbf{R}_+)$$

is w*-continuous, it follows that H_σ is a Borel set in $M(G, \mathcal{A}_+)$.

Now we show that $\mathbf{P}(\partial_e \mathcal{K} \setminus H_\sigma) = 0$. Note that

$$\partial_e \mathcal{K} \setminus H_\sigma = \{\nu \in \partial_e \mathcal{K} : \ \nu \neq \nu * \sigma\} = \bigcup_{n=1}^{\infty} \{\nu \in \partial_e \mathcal{K} : \ \rho_n \nu > \rho_n \nu * \rho_n \sigma\}.$$

Let $\{h_m\}_{m=1}^{\infty}$ be a countable dense set in $K(G, \mathbf{R})_+$. Then

$$\{\nu \in \partial_e \mathcal{K} : \rho_n \nu > \rho_n \nu * \rho_n \sigma\} = \bigcup_{m=1}^{\infty} \{\nu \in \partial_e \mathcal{K} : (\rho_n \nu)(h_m) > (\rho_n \nu * \rho_n \sigma)(h_m)\}.$$

Suppose $\mathbf{P}\{\nu \in \partial_e \mathcal{K} : (\rho_n \nu)(h_m) > (\rho_n \nu * \rho_n \sigma)(h_m)\} > 0$ for some m. Then (5.1) implies

$$\begin{aligned}(\rho_n \mu)(h_m) &= \mu(h_m(\cdot)\rho_n) = c \int_{\partial_e \mathcal{K}} \nu(h_m(\cdot)\rho_n)\, \mathrm{d}\mathbf{P}(\nu) = c \int_{\partial_e \mathcal{K}} (\rho_n \nu)(h_m)\, \mathrm{d}\mathbf{P}(\nu) \\ &> c \int_{\partial_e \mathcal{K}} (\rho_n \nu * \rho_n \sigma)(h_m)\, \mathrm{d}\mathbf{P}(\nu) = (\rho_n \mu * \rho_n \sigma)(h_m) = (\rho_n \mu)(h_m)\end{aligned}$$

which is impossible. Hence we have shown that $\mathbf{P}(\partial_e \mathcal{K} \setminus H_\sigma) = 0$, that is, $\mathbf{P}(\partial_e \mathcal{K} \bigcap H_\sigma) = 1$ where $\partial_e \mathcal{K} \bigcap H_\sigma \subset (\partial C_\sigma \bigcap H_\sigma) \bigcup \{0\} = \partial H_\sigma \bigcup \{0\}$. By absorbing the constant c in (5.1) into ν we have the representation as stated. ∎

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